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Einstein Exponential Operational Laws Based on Fractional Orthotriple Fuzzy Sets and Their Applications in Decision Making Problems

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Abstract: The fractional orthotriple fuzzy set (FOFS) model is a recently created extension of fuzzy sets (FS) for coping with ambiguity in DM. The purpose of this study is to define new exponential and Einstein exponential operational (EO) laws for fractional orthotriple fuzzy sets and the aggregation procedures that accompany them. We present the operational laws for exponential and Einstein exponential FOFSs which have crisp numbers as base values and fractional orthotriple fuzzy numbers as exponents (weights). The proposed operations' qualities and characteristics are then explored. Based on the defined operation laws regulations, various new FOFS aggregation operators, named as fractional orthotriple fuzzy weighted exponential averaging (FOFWEA), fractional orthotriple fuzzy ordered weighted exponential averaging (FOFOWEA), fractional orthotriple fuzzy hybrid weighted averaging (FOFHWEA), fractional orthotriple fuzzy Einstein weighted exponential averaging (FOFEWEA), fractional orthotriple fuzzy Einstein ordered weighted exponential averaging (FOFEOWEA), and fractional orthotriple fuzzy Einstein hybrid weighted exponential averaging (FOFEHWEA) operators are presented. A decision-making algorithm based on the newly defined aggregation operators is proposed and applied to a multicriteria group decision-making (MCGDM) problem related to bank security. Finally, we compare our proposed method with other existing methods.

Keywords: fractional orthotriple fuzzy set; exponential operational laws; Einstein exponential operational laws; aggregation operators; decision making

MSC: 90B50; 91B06; 03E72; 47S40; 03B52

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1. Introduction

The notion of fuzzy set (FS) theory [1] was introduced by Zadeh in 1965 by giving membership value to every element of the group in the range [0, 1], and it is used to describe circumstances where outcomes are inaccurate. This classic FS has been utilized in a wide range of applications, including as DM, clustering analysis, pattern identification, and medicinal treatment. Al-shami et al. [2] defined SR-fuzzy sets with weighted aggregated operators and discussed their application to decision making. Also, Al-shami and Mhemdi [3] suggested generalized frames for orthopair fuzzy sets: (m, n)-fuzzy sets and their applications to multicriteria decision-making methods. Al-shami et al. [4] developed a new generalization of fuzzy soft sets: (a, b)-Fuzzy soft sets. Unluckily, this conventional FS theory only works with positive membership grade components. To address this issue, Atanassov proposed negative membership grade to fill gaps in FS theory, and the resultant collection is recognized as an intuitionistic fuzzy set (IFS) [5]. As a result, IFS theory is a development of FS theory. Atanassov dealt with both MD and NMD, when the total of the two numbers is smaller than or equivalent to one $(P_{\breve{A}} + N_{\breve{A}} \leq 1)$. In rare circumstances, decision-makers may supply values that are favored, such as $P_{\breve{A}} = 0.5$ and $N_{\breve{A}} = 0.7$, which obviously encroach on the IFS criterion because the total of the two numbers is smaller than or equal to one. To address such challenges, Yager [6] created the idea of Pythagorean FS, with the requirement that $P_{\breve{A}}^{\breve{g}} + N_{\breve{A}}^{\breve{g}} \leq 1$.

PFSs definitely control uncertainty more successfully than IFS, making Pythagorean FS theory a more relevant and attractive study subject. Yager and Abbasov [7] proposed many aggregation operations to tackle MCDM issues in a Pythagorean fuzzy environment. Another significant generalization of classical fuzzy sets is the neutrosophic set [8], which is expanded to neutrosophic cubic sets (NCSs) [9]. Many contributions to neutrosophic sets (NSs) and NCSs associated with the operators of aggregation can be found in the literature. Alia et al. [10] discussed NCS theory and used it in pattern recognition. In addition, Je created operations and aggregating techniques for NCSs. Ajay et al. [11] used NCSs for multicriteria decision making (MCDM) using geometric Bonferroni mean operations. Latest, Atta et al. [12] used the concept of NSs for a higher-level picture steganography that relied on modification direction. Gundogdu et al. [13] introduced the purpose of spherical orthotriple FS and its accompanying theory, and this design is among the most recent upgrades to FS hypothesis, which has a triplicate membership pattern consisting of an MF, an NMF, and hesitancy function, and the total of the squares is one or less.

When compared to PFSs, the SFS model can manage uncertainty, imprecision, and ambiguity more efficiently. A recent analysis of some of the most current literature reveals a growing tendency in SFS research. Ashraf et al. [14] created aggregation operator sequences in a spherical fuzzy environment. Ashraf et al. [15] established a gray technique (GRA) dependent upon the innovative notion of spherical linguistic fuzzy Choquet integrals, whereas Jin et al. [16] created and implemented the logarithmic operator for SFSs for decision support systems. Rafiq et al. [17] presented a cosine similarity measure for the spherical fuzzy set model to aid in making decisions in the face of ambiguous and inaccurate data. Ashraf et al. [18] developed a multicriteria group decision-making (MCGDM) approach in the context of the spherical fuzzy environment and used it for a multicriteria group decision-making (MCGDM) issue. Gundogdu et al. [19] adapted the famous VIKOR approach for the SFS model and used it to an MCDM issue in the context of the spherical fuzzy effects. Acharjya and Rathi [20] developed an extensive decision-making technique that combines models of fuzzy rough set with genetic algorithm, which they used for an MCDM problem linked to smart agriculture. Sharaf et al. [21,22] investigated a text overview extraction methodology based on fuzzy logic and suggested a document categorization method based on a fuzzy clustering approach. Gou et al. [23] defined the EO principles for IFSs and presented several recent operators of aggregation for the IFS model, whereas Garg [24] presented recent EO principles for the PFS model and operators of aggregation dependent upon these recently defined EO principles for better handling information ambiguity, impreciseness, and confusion. Furthermore, Borg et al. [25] employed EO laws of PFSs to create projection models for decision making, whereas Haque et al. [26] used the notion from EO principles for broad SFSs.

Akram et al. [27] investigated the purpose of the SF diagram and presented a few findings on the symmetrical difference, dismissal degree, and absolute degree for spherical fuzzy diagrams, whereas Ashraf et al. [28] developed a novel unified technique dependent upon the famous MCDM approaches Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) and Complex Proportional Assignment. Quek et al. [29] created recent operational principles for the T-SFS model and presented Einstein aggregation operators of two types in this model, which they then used on the multiattribute multiperception DM problem involving pollution degree in five major Chinese cities. Shishavan et al. [30] presented the Jaccard, exponential, and square root cosine similarity measures in an FOF scenario and used them to execute those approaches for MCDM issues with medical evaluation and manufacturer choice, while for the SFS model, Aydogdu and Gul [31] presented a unique entropy measurement and evaluated its effectiveness against various other metrics that are currently in use in the literature. While Garg et al. [32] recommended the idea of power AO for the TSFS model and an MCDM algorithm centered on these AO, Ali et al. [33] suggested the novel concept of complex T-SFSs and their operational principles, along with two new operators of aggregation for this model. The T-fractional orthotriple fuzzy soft set model and its AO were proposed by Liu et al. [34], Guleria, and Bajaj [35]. Furthermore, the idea of linguistic T-fractional orthotriple fuzzy numbers was proposed, and two novel MCDM algorithms and a weighted aggregation operator were proposed for this objective.

Regarding the use of SFS statistical frameworks for MCDM techniques, Sharaf and Khalil [36] expanded on the famous MCDM procedure of Tomada De Decisao Interactive Multicriterio (TODIM) to the SF surroundings to allow the decision-makers' hesitation degree to be described properly, while Mathew et al. [37] suggested an unusual model for making decisions that integrates the famous Kahraman and Gundogdu established notion of interval worth SFSs (IV-SFS). They described several crucial supporting notions for this structure, such as the score and accuracy values, as well as the mean arithmetic and geometric operations, in [38]. The writers then proposed an IV-SFS dependent TOPSIS approach and used it to obtain MCDM issues connected to the option of 3D printers. Barukab et al. [39] defined an extended measure of distance for SFSs dependent on spherical orthotriple; the fuzzy entropy was used to calculate the undetermined values of the criterion. Farrokhizadeh et al. [40] extended this to apply the original maximized variability approach to the fractional orthotriple fuzzy atmosphere with only one value and a range of SFSs toascertain. Akram et al. [41] suggested four novel AOs for the complex SFS designs and applied them to expand the multicriteria improvement and compromising solution (VIKOR) approach to the complex fractional orthotriple fuzzy surroundings, while Ali et al. [42] presented a TOPSIS technique centered on a complex SFS model along with two kinds of Bonferroni mean operators of aggregation.

Motivated by the abovementioned fast-moving studies on FOFS theory, the primary purpose of this study is to establish exponential and Einstein exponential operating rules for the FOFS model, resulting in conclusions in the prolonged closed range of [1, 2], along with building exponential FOFWEA, FOFOEA, FOFHWEA, and Einstein exponential fractional orthotriple fuzzy aggregation operators FOFEWEA, FOFEOWEA, and FOFEHWEA.

The rest of the paper is structured as follows. Section 2 offers a summary of the crucial fundamental information relevant to the FOFS model and its operations, operational laws, and AO. Section 3 introduces the EO rules for the FOFS model, as well as the accompanying operations and attributes. Following that, new exponential farctional orthotriple fuzzy aggregation operations are introduced. Section 4 defines the Einstein exponential operation principles for the FOFS model and their accompanying operations and attributes, as well as the Einstein exponential fractional orthotriple fuzzy AO. Section 5 presents an MCDM method dependent upon the newly developed fractional orthotriple fuzzy operators of aggregation. In Section 6, the new relationships of technology utilization and utility fractional orthotriple fuzzy aggregation operators are demonstrated through their use in an MCDM problem involving the ranking of various ratings of several methods of psychotherapy for emotional issues encountered by children. A comparison study is provided in Section 7 to validate the suggested MCDM approach, in which the outcomes obtained utilizing our suggested strategy are compared with the present literary results acquired utilizing fuzzy-based MCDM approaches. Section 8 contains the conclusion of the paper.

2. Preliminaries

This section will discuss some fundamental ideas of fractional orthotriple fuzzy sets.

Definition 1 ([1]). *An FS F is defined on a conversation universe U as the shape*

$$F = \{\langle P_F(\acute{r})\rangle | \acute{r} \in U\}, \tag{1}$$

where $P_F(\acute{r}) :\to [0,1]$. Here, $P_F(\acute{r})$ represents the MF for every $\acute{r}$.

Definition 2 ([5])**.** *An IFS $\breve{A}$ is defined as a collection of ordered pairs provided on a universal collection,*

$$\breve{A} = \{\langle \acute{r}, (P_{\breve{A}}(\acute{r}), N_{\breve{A}}(\acute{r}))\rangle | \acute{r} \in U\}, \tag{2}$$

where $P_{\breve{A}}(\acute{r}) : U \to [0,1], N_{\breve{A}}(\acute{r}) : U \to [0,1]$ and satisfy the condition $P_{\breve{A}}(\acute{r}) + N_{\breve{A}}(\acute{r}) \leq 1$ for every component $\acute{r} \in U$. Here, the MF and NMFs are denoted as $P_{\breve{A}}(\acute{r})$ and $N_{\breve{A}}(\acute{r})$, correspondingly.

Definition 3 ([43])**.** *Let $\breve{A}_{\hat{s}}$ be an FOFS in the conversation universe U be defined by*

$$\breve{A}_{\hat{s}} = \left\{\left\langle \acute{r}, (P_{\breve{A}_{\hat{s}}}(\acute{r})), N_{\breve{A}_{\hat{s}}}(\acute{r}), I_{\breve{A}_{\hat{s}}}(\acute{r})\right\rangle | \acute{r} \in U\right\}, \tag{3}$$

where $P_{A_{\hat{s}}}(\acute{r}) : U \to [0,1], N_{\breve{A}_{\hat{s}}}(\acute{r}) : U \to [0,1]$ and $0 \leq P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})) + N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) + I^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) \leq 1$ for each $\acute{r}$, the values $P_{\breve{A}_{\hat{s}}}(\acute{r})), N_{\breve{A}_{\hat{s}}}(\acute{r}), I_{\breve{A}_{\hat{s}}}(\acute{r})$ are MF, NMF, and hesitancy function of $\acute{r}$ in $\breve{A}_{\hat{s}}$, defined as

$$\pi_{\breve{A}_{\hat{s}}}(\acute{r}) = \sqrt[\breve{g}]{1 - \left(P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}), N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}), I^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)}$$

Definition 4. *The score function and accuracy function of the fractional orthotriple FS are defined correspondingly as*

$$\hat{s}(\breve{A}_{\hat{s}}(\acute{r})) = P^{f}_{\breve{A}_{\hat{s}}}(\acute{r}) - N^{f}_{\breve{A}_{\hat{s}}}(\acute{r}) - I^{f}_{A_{\hat{s}}}(\acute{r}) \tag{4}$$

and

$$\breve{A}(\breve{A}_{\hat{s}}(\acute{r})) = P^{f}_{\breve{A}_{\hat{s}}}(\acute{r}) + N^{f}_{\breve{A}_{\hat{s}}}(\acute{r}) + I^{f}_{\breve{A}_{\hat{s}}}(\acute{r}) \tag{5}$$

Definition 5. *The fundamental activities of fractional orthotriple fuzzy numbers are defined as*

1. $$\breve{A}_{\hat{s}} \oplus B_{\hat{s}} = \left\{\begin{array}{c} \sqrt[\breve{g}]{P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) + P^{\breve{g}}_{B_{\hat{s}}}(\acute{r}) - P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})}, N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})N^{\breve{g}}_{B_{\hat{s}}}(\acute{r}), \\ \sqrt[\breve{g}]{\left(1 - P^{\breve{g}}_{B_{\hat{s}}}(\acute{r})\right)I^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) + \left(1 - P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)I^{\breve{g}}_{B_{\hat{s}}}(\acute{r}) - I^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})I^{\breve{g}}_{B_{\hat{s}}}(\acute{r})} \end{array}\right\};$$
2. $$\breve{A}_{\hat{s}} \otimes B_{\hat{s}} = \left\{\begin{array}{c} P_{\breve{A}_{\hat{s}}}(\acute{r})P_{B_{\hat{s}}}(\acute{r}), \sqrt[\breve{g}]{(N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) + (N^{\breve{g}}_{B_{\hat{s}}}(\acute{r}) - (N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}).(N^{\breve{g}}_{B_{\hat{s}}}(\acute{r})}, \\ \sqrt[\breve{g}]{\left(1 - (N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)I^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) + \left(1 - (N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)I^{\breve{g}}_{B_{\hat{s}}}(\acute{r}) - I^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})I^{\breve{g}}_{B_{\hat{s}}}(\acute{r})} \end{array}\right\};$$
3. $$\epsilon.\breve{A}_{\hat{s}} = \left\{\begin{array}{c} \sqrt[\breve{g}]{1 - \left(1 - P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)^{\epsilon}}, N^{\epsilon}_{\breve{A}_{\hat{s}}}(\acute{r}), \\ \sqrt[\breve{g}]{\left(1 - P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)^{\epsilon} - \left(1 - P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) - I^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)^{\epsilon}} \end{array}\right\}, \epsilon > 0$$
4. $$\breve{A}_{\hat{s}} = \left\{\begin{array}{c} P^{\epsilon}_{\breve{A}_{\hat{s}}}(\acute{r}), \sqrt[\breve{g}]{1 - \left(1 - N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)^{\epsilon}}, \\ \sqrt[\breve{g}]{\left(1 - N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)^{\epsilon} - \left(1 - N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) - I^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r})\right)^{\epsilon}} \end{array}\right\}, \epsilon > 0$$

3. Exponential Operational Laws of FOFSs

This section defines new EO principles of FOFSs and their operations.

Definition 6. *Let U be the conversation universe, and $\beta_{\hat{s}} = (P_{\beta_{\hat{s}}}, N_{\beta_{\hat{s}}}, I_{\beta_{\hat{s}}})$ be a fractional orthotriple fuzzy number (FOFN); then, the exponential operations of $\beta_{\hat{s}}$ is defined as*

$$\epsilon^{\beta_{\hat{s}}} = \left\{\begin{array}{l} \left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}}}}}, \sqrt[\breve{g}]{1 - \epsilon^{\breve{g}N_{\beta_{\hat{s}}}}}, \sqrt[\breve{g}]{1 - \epsilon^{\breve{g}I_{\beta_{\hat{s}}}}}\right); \epsilon \in (0,1) \\ \left((\frac{1}{\epsilon})^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}}}}}, \sqrt[\breve{g}]{1 - (\frac{1}{\epsilon})^{\breve{g}N_{\beta_{\hat{s}}}}}, \sqrt[\breve{g}]{1 - (\frac{1}{\epsilon})^{\breve{g}I_{\beta_{\hat{s}}}}}\right); \epsilon \geq 1 \end{array}\right. \tag{6}$$

Theorem 1. *For any FOFN $\beta_{\hat{s}}$, the value of $\epsilon^{\beta_{\hat{s}}}$ is an FOFN in the prolonged range $[1,2]$.*

Proof. Let $\beta_{\hat{s}} = (P_{\beta_{\hat{s}}}, N_{\beta_{\hat{s}}}, I_{\beta_{\hat{s}}})$ be an FOFN, where $P_{\beta_{\hat{s}}}, N_{\beta_{\hat{s}}}$ and $I_{\beta_{\hat{s}}}$belong to $[0,1]$ with the condition that $0 \leq P_{\beta_{\hat{s}}} + N_{\beta_{\hat{s}}} + I_{\beta_{\hat{s}}} \leq 1$ $\square$

Case 1. Let $\epsilon \in (0,1)$, then the values of $\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}}}}}$, $\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\beta_{\hat{s}}}}}$, and $\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\beta_{\hat{s}}}}}$ lie in $[0,1]$, meeting the requirement that $1 \leq (\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}}}}})^{\breve{g}} + \left(\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\beta_{\hat{s}}}}}\right)^{\breve{g}} + \left(\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\beta_{\hat{s}}}}}\right)^{\breve{g}} \leq 2$.

Case 2. When $\epsilon \geq 1$ and $0 \leq \frac{1}{\epsilon} \leq 1$, it is self-evident that $\epsilon^{\beta_{\hat{s}}}$ is an FOFN. As a result of the two situations, the values of $\epsilon^{\beta_{\hat{s}}}$ are FOFNs in the prolonged range $[1,2]$.

For example, let $\beta_{\hat{s}} = (0.73, 0.28, 0.32)$ be an FOFN and $\epsilon = 0.82$. Then,

$$\begin{aligned} \epsilon^{\beta_{\hat{s}}} &= 0.82^{(0.73,0.28,0.32)}, 0.82^{(0.73,0.28,0.32)} \\ &= (0.82^{\sqrt[3]{1-0.73^3}}, \sqrt[3]{1-0.82^{3*0.28}}, \sqrt[3]{1-0.82^{3*0.32}}) \\ &= (0.6958, 0.5354, 0.5577) \end{aligned}$$

If

$$\begin{aligned} \epsilon &= \left\langle \left(\frac{1}{4}\right)^{\beta_{\hat{s}}} = \left(\frac{1}{4}\right)^{\sqrt[3]{1-0.73^3}}, \sqrt[3]{1-(\frac{1}{4})^{3*0.28}}, \sqrt[3]{1-(\frac{1}{4})^{3*0.32}} \right\rangle \\ &= (0.3877, 0.7348, 0.7669) \end{aligned}$$

Furthermore, we list some of the fundamental operations on $\epsilon^{\beta_{\hat{s}}}$.

Definition 7. *Let $\beta_{\hat{s}_1}$ and $\beta_{\hat{s}_2}$ be two fractional orthotriple fuzzy numbers. Then the fundamental exponential operational principles are mentioned, following:*

1. $$\epsilon^{\beta_{\hat{s}_1}} \oplus \epsilon^{\beta_{\hat{s}_2}} = \left\{ \left\langle \begin{array}{c} \sqrt[\breve{g}]{1-\left(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)\left((1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}})\right)}, \\ \sqrt[\breve{g}]{\left(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}\right)\left(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}\right)}, \\ \sqrt[\breve{g}]{\begin{array}{c}\left(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)\left(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}}\right) \\ -\left(\epsilon^{\breve{g}I_{\beta_{\hat{s}_1}}}-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)\left(\epsilon^{\breve{g}I_{\beta_{\hat{s}_2}}}-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}}\right)\end{array}} \end{array} \right\rangle \right\};$$

2. $$\epsilon^{\beta_{\hat{s}_1}} \otimes \epsilon^{\beta_{\hat{s}_2}} = \left\{ \left\langle \begin{array}{c} \epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}+\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}}, \sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}}, \\ \sqrt[\breve{g}]{\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}-(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})} \end{array} \right\rangle \right\};$$

3. $$K\lambda^{\beta_{\hat{s}}} = \left\{ \left\langle \begin{array}{c} \sqrt[\breve{g}]{1-\left(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)^K}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}}}})^K}, \\ \sqrt[\breve{g}]{\left(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)^K-\left(\epsilon^{\breve{g}I_{\beta_{\hat{s}}}}-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)^K} \end{array} \right\rangle \right\};$$

4. $$[\epsilon^{\beta_{\hat{s}}}]^K = \left\{ \left\langle \begin{array}{c} \epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}}}}})^K, \sqrt[\breve{g}]{1-(\epsilon^{\breve{g}N_{\beta_{\hat{s}}}})^K}, \\ \sqrt[\breve{g}]{(\epsilon^{\breve{g}N_{\beta_{\hat{s}}}})^K-(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}}}}-\epsilon^{\breve{g}I_{\beta_{\hat{s}}}})^K} \end{array} \right\rangle \right\}, \forall K > 0.$$

Theorem 2. *Let $\beta_{\hat{s}_1} = (P_{\beta_{\hat{s}_1}}, N_{\beta_{\hat{s}_1}}, I_{\beta_{\hat{s}_1}})$ and $\beta_{\hat{s}_2} = (P_{\beta_{\hat{s}_2}}, N_{\beta_{\hat{s}_2}}, I_{\beta_{\hat{s}_2}})$ be fractional orthotriple fuzzy numbers (FOFNs) and $\epsilon \in (0,1)$. Then, the following holds:*

(1) $\epsilon^{\beta_{\hat{s}_1}} \oplus \epsilon^{\beta_{\hat{s}_2}} = \epsilon^{\beta_{\hat{s}_2}} \oplus \epsilon^{\beta_{\hat{s}_1}}$

(2) $\epsilon^{\beta_{\hat{s}_1}} \otimes \epsilon^{\beta_{\hat{s}_2}} = \epsilon^{\beta_{\hat{s}_2}} \otimes \epsilon^{\beta_{\hat{s}_1}}$

Proof. The proof of Theorem 2 is straightforward from Definition 3. □

Theorem 3. *Let $\beta_{\hat{s}_1} = (P_{\beta_{\hat{s}_1}}, N_{\beta_{\hat{s}_1}}, I_{\beta_{\hat{s}_1}})$ for $(p = 1,2,3)$ be three fractional orthotriple fuzzy numbers (FOFNs) and $\epsilon \in (0,1)$. Then, the following holds:*

(1). $(\epsilon^{\beta_{\hat{s}_1}} \oplus \epsilon^{\beta_{\hat{s}_2}}) \oplus \epsilon^{\beta_{\hat{s}_3}} = \epsilon^{\beta_{\hat{s}_1}} \oplus (\epsilon^{\beta_{\hat{s}_2}} \oplus \epsilon^{\beta_{\hat{s}_3}})$

(2). $(\epsilon^{\beta_{\hat{s}_1}} \otimes \epsilon^{\beta_{\hat{s}_2}}) \otimes \epsilon^{\beta_{\hat{s}_3}} = \epsilon^{\beta_{\hat{s}_1}} \otimes (\epsilon^{\beta_{\hat{s}_2}} \otimes \epsilon^{\beta_{\hat{s}_3}})$

Proof. The proof of Theorem 3 is straightforward from Definition 3. □

Theorem 4. *Let $\beta_{\hat{s}_1} = (P_{\beta_{\hat{s}_1}}, N_{\beta_{\hat{s}_1}}, I_{\beta_{\hat{s}_1}})$ and $\beta_{\hat{s}_2} = (P_{\beta_{\hat{s}_2}}, N_{\beta_{\hat{s}_2}}, I_{\beta_{\hat{s}_2}})$ be fractional orthotriple fuzzy numbers (FOFNs); $K, K_1, K_2 > 0$, be three real numbers; and $\epsilon, \epsilon_1, \epsilon_2 \in (0,1)$. Then, the following holds:*

(1). $K(\epsilon^{\beta_{\hat{s}_1}} \oplus \epsilon^{\beta_{\hat{s}_2}}) = K(\epsilon^{\beta_{\hat{s}_1}}) \oplus K(\epsilon^{\beta_{\hat{s}_2}})$;

(2). $(\epsilon^{\beta_{\hat{s}_1}} \otimes \epsilon^{\beta_{\hat{s}_2}})^K = (\epsilon^{\beta_{\hat{s}_1}})^K \otimes (\epsilon^{\beta_{\hat{s}_2}})^K$;

(3). $K_1\epsilon^{\beta_{\hat{s}_1}} \oplus K_2\epsilon^{\beta_{\hat{s}_2}} = (K_1 + K_2)\epsilon^{\beta_{\hat{s}_1}}$;

(4). $(\epsilon^{\beta_{\hat{s}_1}})^{K_1} \otimes (\epsilon^{\beta_{\hat{s}_2}})^{K_2} = (\epsilon^{\beta_{\hat{s}_1}})^{K_1+K_2}$;

(5). $(\epsilon_1)^{\beta_{\hat{s}_1}} \otimes (\epsilon_2)^{\beta_{\hat{s}_2}} = (\epsilon_1\epsilon_2)^{\beta_{\hat{s}_1}}$.

Proof. For two fractional orthotriple fuzzy numbers $\beta_{\hat{s}_1}$ and $\beta_{\hat{s}_2}$, by Definition 3, we obtain

$$\epsilon^{\beta_{\hat{s}_1}} = \left((\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}, \sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}}, \sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\beta_{\hat{s}_1}}}}) \right), \tag{7}$$

$$\epsilon^{\beta_{\hat{s}_2}} = \left((\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}}, \sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}}, \sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\beta_{\hat{s}_2}}}}) \right) \tag{8}$$

and so by employing the EO principles specified in the Definition 4, we obtain

$$\epsilon^{\beta_{\hat{s}_1}} \oplus \epsilon^{\beta_{\hat{s}_2}} = \left\{ \left\langle \begin{array}{c} \sqrt[\breve{g}]{1-\left(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)\left(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}}\right)}, \\ \sqrt[\breve{g}]{\left(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}\right)\left(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}\right)}, \\ \hline \sqrt[\breve{g}]{\begin{array}{c} \left(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)\left(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}}\right) \\ -\left(\epsilon^{\breve{g}I_{\beta_{\hat{s}_1}}}-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)\left(\epsilon^{\breve{g}I_{\beta_{\hat{s}_2}}}-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}}\right) \end{array}} \end{array} \right\rangle \right\} \tag{9}$$

$$\epsilon^{\beta_{\hat{s}_1}} \otimes \epsilon^{\beta_{\hat{s}_2}} = \left\{ \left\langle \frac{\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}+\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}}, \sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}},}{\sqrt[\breve{g}]{\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}-(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})}} \right\rangle \right\}. \tag{10}$$

(1) For a real number $K > 0$, we have

$$K(\epsilon^{\beta_{\hat{s}_1}} \oplus \epsilon^{\beta_{\hat{s}_2}}) = \left\{ \begin{array}{c} \sqrt[\breve{g}]{1-(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^K(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}})^K}, \\ \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^K(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^K}, \\ \sqrt[\breve{g}]{\begin{array}{c} (1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^K(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}})^K- \\ (\epsilon^{\breve{g}I_{\beta_{\hat{s}_1}}}-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^K(\epsilon^{\breve{g}I_{\beta_{\hat{s}_2}}}-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}})^K \end{array}} \end{array} \right\}$$

$$\begin{aligned} &= \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt[\breve{g}]{1-(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^K}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^K}, \\ \sqrt[\breve{g}]{(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^K-(\epsilon^{\breve{g}I_{\beta_{\hat{s}_1}}}-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})} \end{array} \right) \\ \oplus \left(\begin{array}{c} \sqrt[\breve{g}]{1-(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}})^K}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^K}, \\ \sqrt[\breve{g}]{(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}})^K-(\epsilon^{\breve{g}I_{\beta_{\hat{s}_2}}}-\epsilon^{\breve{g}}\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}})} \end{array} \right) \end{array} \right\} \\ &= K(\epsilon^{\beta_{\hat{s}_1}}) \oplus K(\epsilon^{\beta_{\hat{s}_2}}) \end{aligned}$$

(2) For two fractional orthotriple fuzzy numbers $\beta_{\hat{s}_1}$ and $\beta_{\hat{s}_2}$, and a real number $K > 0$, we have

$$\begin{aligned} &(\epsilon^{\beta_{\hat{s}_1}} \otimes \epsilon^{\beta_{\hat{s}_2}})^K \\ &\left\{ \begin{array}{c} \left(\left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)^K \left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}}\right)^K, \sqrt[\breve{g}]{1-(\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^K(\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^K} \right), \\ \left(\sqrt[\breve{g}]{(\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^K(\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^K-(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^K(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^K} \right) \end{array} \right\}; \\ &\epsilon \quad > \quad 0 \end{aligned}$$

$$\begin{aligned} (\epsilon^{\beta_{\hat{s}_1}} \otimes \epsilon^{\beta_{\hat{s}_2}})^K \quad &= \left\{ \begin{array}{c} \left(\begin{array}{c} \left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)^K, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^K}, \\ \sqrt[\breve{g}]{(\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^K-(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}}-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^K} \end{array} \right) \\ \otimes \left(\begin{array}{c} \left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{s2}}}}\right)^K, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^K}, \\ \sqrt[\breve{g}]{(\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^K-(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}}-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^K} \end{array} \right) \end{array} \right\} \\ &= (\epsilon^{\beta_{\hat{s}_1}})^K \otimes (\epsilon^{\beta_{\hat{s}_2}})^K \end{aligned}$$

(3) For a fractional orthotriple fuzzy number $\beta_{\hat{s}_1} = (P_{\beta_{\hat{s}_1}}, N_{\beta_{\hat{s}_1}}, I_{\beta_{\hat{s}_1}})$, and a real number $K_1, K_2 > 0$,

$$\begin{aligned}
K_1\epsilon^{\beta_{\hat{s}_1}} \oplus K_2\epsilon^{\beta_{\hat{s}_2}} &= \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt[\breve{g}]{1-(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^{K_1}}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1}}, \\ \sqrt[\breve{g}]{(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^{K_1} - (\epsilon^{\breve{g}I_{\beta_{\hat{s}_1}}} - \epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^{K_1})} \end{array} \right) \\ \oplus \left(\begin{array}{c} \sqrt[\breve{g}]{1-(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}})^{K_2}}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^{K_2}}, \\ \sqrt[\breve{g}]{(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}})^{K_2} - (\epsilon^{\breve{g}I_{\beta_{\hat{s}_2}}} - \epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_2}}}})^{K_2}} \end{array} \right) \end{array} \right\} \\
&= \left\{ \begin{array}{c} (\sqrt[\breve{g}]{1-(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^{K_1+K_2}}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1+K_2}}, \\ \sqrt[\breve{g}]{(1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^{K_1+K_2} - (\epsilon^{\breve{g}I_{\beta_{\hat{s}_1}}} - \epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}})^{K_1+K_2}} \end{array} \right\} \\
&= (K_1+K_2)\epsilon^{\beta_{\hat{s}_1}}
\end{aligned}$$

(4) For a fractional orthotriple fuzzy number $\beta_{\hat{s}_1} = (P_{\beta_{\hat{s}_1}}, N_{\beta_{\hat{s}_1}}, I_{\beta_{\hat{s}_1}})$, and a real number $K_1, K_2 > 0$,

$$(\epsilon^{\beta_{\hat{s}_1}})^{K_1} = \left\{ \begin{array}{c} \left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)^{K_1}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1}}, \\ \sqrt[\breve{g}]{(\epsilon^{\breve{g}I_{\beta_{\hat{s}_1}}})^{K_1} - (1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}} - \epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1}} \end{array} \right\}, \forall K_1 > 0.$$

$$(\epsilon^{\beta_{\hat{s}_2}})^{K_2} = \left\{ \begin{array}{c} \left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{s2}}}}\right)^{K_2}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^{K_2}}, \\ \sqrt[\breve{g}]{(\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^{K_2} - (1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}} - \epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^{K_2}} \end{array} \right\}, \forall K_2 > 0.$$

$$\begin{aligned}
(\epsilon^{\beta_{\hat{s}_1}})^{K_1} \otimes (\epsilon^{\beta_{\hat{s}_2}})^{K_2} &= \left\{ \begin{array}{c} \left(\begin{array}{c} \left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)^{K_1}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1}}, \\ \sqrt[\breve{g}]{(\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1} - (1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}} - \epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1}} \end{array} \right) \\ \otimes \left(\begin{array}{c} \left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{s2}}}}\right)^{K_2}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^{K_2}}, \\ \sqrt[\breve{g}]{(\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^{K_2} - (1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}} - \epsilon^{\breve{g}N_{\beta_{\hat{s}_2}}})^{K_2}} \end{array} \right) \end{array} \right\} \\
&= \left\{ \begin{array}{c} \left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}\right)^{K_1+K_2}, \sqrt[\breve{g}]{(1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1+K_2}}, \\ \sqrt[\breve{g}]{(\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1+K_2} - (1-\epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}} - \epsilon^{\breve{g}N_{\beta_{\hat{s}_1}}})^{K_1+K_2}} \end{array} \right\} \\
&= (\epsilon^{\beta_{\hat{s}_1}})^{K_1+K_2}
\end{aligned}$$

(5). For a fractional orthotriple fuzzy number $\beta_{\hat{s}_1} = (P_{\beta_{\hat{s}_1}}, N_{\beta_{\hat{s}_1}}, I_{\beta_{\hat{s}_1}})$, and a real number $\epsilon_1, \epsilon_2 > 0$,

$$
\begin{aligned}
(\epsilon_1)^{\beta_{\hat{s}_1}} \otimes (\epsilon_2)^{\beta_{\hat{s}_2}} &= \left\{ \begin{array}{c} ((\epsilon_1)^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}, \sqrt[\breve{g}]{1-\epsilon_1^{N^{\breve{g}}_{\beta_{\hat{s}_1}}}}, \sqrt[\breve{g}]{1-\epsilon_1^{I^{\breve{g}}_{\beta_{\hat{s}_1}}}}) \\ \otimes((\epsilon_2)^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}, \sqrt[\breve{g}]{1-\epsilon_2^{N^{\breve{g}}_{\beta_{\hat{s}_1}}}}, \sqrt[\breve{g}]{1-\epsilon_2^{I^{\breve{g}}_{\beta_{\hat{s}_1}}}}) \end{array} \right\} \\
&= \left\{ \begin{array}{c} (\epsilon_1\epsilon_2)^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\beta_{\hat{s}_1}}}}, \sqrt[\breve{g}]{1-(\epsilon_1^{N^{\breve{g}}_{\beta_{\hat{s}_1}}})(\epsilon_2^{N^{\breve{g}}_{\beta_{\hat{s}_1}}})}, \\ \sqrt[\breve{g}]{\epsilon_1^{\breve{g}N_{\beta_{\hat{s}_1}}}\epsilon_2^{\breve{g}N_{\beta_{\hat{s}_1}}} - (1-\epsilon_1^{\breve{g}N_{\beta_{\hat{s}_1}}} - \epsilon_1^{\breve{g}I_{\beta_{\hat{s}_1}}})(1-\epsilon_2^{\breve{g}N_{\beta_{\hat{s}_1}}} - \epsilon_2^{\breve{g}I_{\beta_{\hat{s}_1}}})} \end{array} \right\} \\
&= (\epsilon_1\epsilon_2)^{\beta_{\hat{s}_1}}
\end{aligned}
$$

□

Theorem 5. *Let $\beta_{\hat{s}} = (P_{\beta_{\hat{s}}}, N_{\beta_{\hat{s}}}, I_{\beta_{\hat{s}}})$ be a fractional orthotriple fuzzy number and $\epsilon_1, \epsilon_2 > 0$. When $\epsilon_1 \geq \epsilon_2$, we can obtain $(\epsilon_1)^{\beta_{\hat{s}}} \geq (\epsilon_2)^{\beta_{\hat{s}}}$ for $\epsilon_1, \epsilon_2 \in (0,1)$ and $(\epsilon_1)^{\beta_{\hat{s}}} \leq (\epsilon_2)^{\beta_{\hat{s}}}$ for $\epsilon_1, \epsilon_2 \geq 1$.*

Proof. If $\epsilon_1 \geq \epsilon_2$ and $\epsilon_1, \epsilon_2 \in (0,1)$, then, using FOFN's EO rules, we obtain

$$(\epsilon_1)^{\beta_{\hat{s}}} = \{((\epsilon_1)^{\sqrt[\breve{g}]{1-(P^{\breve{g}}_{\beta_{\hat{s}}})}}, \sqrt[\breve{g}]{1-\epsilon_1^{N^{\breve{g}}_{\beta_{\hat{s}}}}}, \sqrt[\breve{g}]{1-\epsilon_1^{I^{\breve{g}}_{\beta_{\hat{s}}}}})\} \tag{11}$$

$$(\epsilon_2)^{\beta_{\hat{s}}} = \{((\epsilon_2)^{\sqrt[\breve{g}]{1-(P^{\breve{g}}_{\beta_{\hat{s}}})}}, \sqrt[\breve{g}]{1-\epsilon_2^{N^{\breve{g}}_{\beta_{\hat{s}}}}}, \sqrt[\breve{g}]{1-\epsilon_2^{I^{\breve{g}}_{\beta_{\hat{s}}}}})\} \tag{12}$$

The score values of $(\epsilon_1)^{\beta_{\hat{s}}}$ and $(\epsilon_2)^{\beta_{\hat{s}}}$ are indicated as follows: $\hat{s}((\epsilon_1)^{\beta_{\hat{s}}})$ and $\hat{s}((\epsilon_2)^{\beta_{\hat{s}}})$ are defined by

$$\hat{s}((\epsilon_1)^{\beta_{\hat{s}}}) = \left((\epsilon_1)^{\sqrt[\breve{g}]{1-(P^{\breve{g}}_{\beta_{\hat{s}}})}} - \sqrt[\breve{g}]{1-\epsilon_1^{I^{\breve{g}}_{\beta_{\hat{s}}}}}\right)^{\breve{g}} - \left(\sqrt[\breve{g}]{1-\epsilon_1^{N^{\breve{g}}_{\beta_{\hat{s}}}}} - \sqrt[\breve{g}]{1-\epsilon_1^{I^{\breve{g}}_{\beta_{\hat{s}}}}}\right)^{\breve{g}}$$

$$\hat{s}((\epsilon_2)^{\beta_{\hat{s}}}) = \left((\epsilon_2)^{\sqrt[\breve{g}]{1-(P^{\breve{g}}_{\beta_{\hat{s}}})}} - \sqrt[\breve{g}]{1-\epsilon_2^{I^{\breve{g}}_{\beta_{\hat{s}}}}}\right)^{\breve{g}} - \left(\sqrt[\breve{g}]{1-\epsilon_2^{N^{\breve{g}}_{\beta_{\hat{s}}}}} - \sqrt[\breve{g}]{1-\epsilon_2^{I^{\breve{g}}_{\beta_{\hat{s}}}}}\right)^{\breve{g}}$$

The values of membership $\beta_{\hat{s}} \in [0,1]$, which indicates that the values of membership grades $P_{\beta_{\hat{s}}}$, the nonmembership grades $N_{\beta_{\hat{s}}}$, and the hesistancy grades $I_{\beta_{\hat{s}}}$ lie in $[0,1]$.

Since $\epsilon_1 \geq \epsilon_2, (\epsilon_1)^{\sqrt[\breve{g}]{1-(P^{\breve{g}}_{\beta_{\hat{s}}})}} \geq (\epsilon_2)^{\sqrt[\breve{g}]{1-(P^{\breve{g}}_{\beta_{\hat{s}}})}}, 1-\epsilon_1^{N^{\breve{g}}_{\beta_{\hat{s}}}} \leq 1-\epsilon_2^{N^{\breve{g}}_{\beta_{\hat{s}}}}$, and $1-\epsilon_1^{I^{\breve{g}}_{\beta_{\hat{s}}}} \leq 1-\epsilon_2^{I^{\breve{g}}_{\beta_{\hat{s}}}}$, then $\hat{s}((\epsilon_1)^{\beta_{\hat{s}}}) \geq \hat{s}((\epsilon_2)^{\beta_{\hat{s}}})$. The following two scenarios arise:

(1) If $\hat{s}((\epsilon_1)^{\beta_{\hat{s}}}) > \hat{s}((\epsilon_2)^{\beta_{\hat{s}}})$, then $(\epsilon_1)^{\beta_{\hat{s}}} > (\epsilon_2)^{\beta_{\hat{s}}}$.

(2) If $\hat{s}((\epsilon_1)^{\beta_{\hat{s}}}) = \hat{s}((\epsilon_2)^{\beta_{\hat{s}}})$, then $(\epsilon_1)^{\sqrt[\breve{g}]{1-(P^{\breve{g}}_{\beta_{\hat{s}}})}} = (\epsilon_2)^{\sqrt[\breve{g}]{1-(P^{\breve{g}}_{\beta_{\hat{s}}})}}, 1-\epsilon_1^{N^{\breve{g}}_{\beta_{\hat{s}}}} = 1-\epsilon_2^{N^{\breve{g}}_{\beta_{\hat{s}}}}$ and $1-\epsilon_1^{I^{\breve{g}}_{\beta_{\hat{s}}}} = 1-\epsilon_2^{I^{\breve{g}}_{\beta_{\hat{s}}}}$, which implies that $H((\epsilon_1)^{\beta_{\hat{s}}}) = H((\epsilon_2)^{\beta_{\hat{s}}})$ and, hence, $(\epsilon_1)^{\beta_{\hat{s}}} = (\epsilon_2)^{\beta_{\hat{s}}}$.

Thus, by combining these two cases, we obtain $(\epsilon_1)^{\beta_{\hat{s}}} \geq (\epsilon_2)^{\beta_{\hat{s}}}$. Suppose that $\epsilon_1, \epsilon_2 \geq 1$ and $\epsilon_1 \geq \epsilon_2$, then we obtain $0 \leq \frac{1}{\epsilon_1} \leq \frac{1}{\epsilon_2} \leq 1$. Similarly, we can obtain $(\epsilon_1)^{\beta_{\hat{s}}} \leq (\epsilon_2)^{\beta_{\hat{s}}}$. □

Exponential Aggregation Operator for FOFNs

Definition 8. *Let* $\beta_{\hat{s}_1} = (P_{\beta_{\hat{s}_1}}, N_{\beta_{\hat{s}_1}}, I_{\beta_{\hat{s}_1}})$ *be a collection of fractional orthotriple fuzzy number and* $\epsilon_p, (p = 1, 2, ..., n)$ *be a grouping of actual values; then,* $FOFWEA : \hat{s}^n \to \hat{s}$*, is known as the fractional orthotriple fuzzy weighted exponential averaging operator, mentioned below:*

$$FOFWEA(\epsilon_p^{\beta_{\hat{s}_1}}, \epsilon_p^{\beta_{\hat{s}_2}}, ..., \epsilon_p^{\beta_{\hat{s}_n}}) = \epsilon_p^{\beta_{\hat{s}_1}} \otimes \epsilon_p^{\beta_{\hat{s}_2}} \otimes ... \epsilon_p^{\beta_{\hat{s}_n}} \tag{13}$$

where $\hat{s}$ *is the grouping of FOFNs and* $\beta_{\hat{s}_p}$ *are the exponential weights of* $\epsilon_p(p = 1, 2, ..., n)$.

Theorem 6. *Let* $\beta_{\hat{s}_p} = (P_{\beta_{\hat{s}_p}}, N_{\beta_{\hat{s}_p}}, I_{\beta_{\hat{s}_p}})$ *be a grouping of FOFNs. The total value obtained by applying the FOFWEA operator also serves as a type of FOFN in the prolonged range* $[1, 2]$*, where*

$$\begin{aligned} & FOFWEA(\beta_{\hat{s}_1}, \beta_{\hat{s}_2}, ..., \beta_{\hat{s}_n}) \\ = & \left\{ \begin{array}{c} (\Pi_{p=1}^n \epsilon_1^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \Pi_{p=1}^n \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}}}, \\ \sqrt[\breve{g}]{\Pi_{p=1}^n \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \Pi_{p=1}^n (1 - \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \epsilon_p^{\breve{g} I_{\beta_{\hat{s}_p}}})}); \ \epsilon_p \in (0, 1); \\ (\Pi_{p=1}^n (\frac{1}{\epsilon_p})^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_p}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \Pi_{p=1}^n (\frac{1}{\epsilon_p})^{\breve{g} N_{\beta_{\hat{s}_p}}}}, \\ \sqrt[\breve{g}]{\Pi_{p=1}^n (\frac{1}{\epsilon_p})^{\breve{g} N_{\beta_{\hat{s}_p}}} - \Pi_{p=1}^n (1 - (\frac{1}{\epsilon_p})^{\breve{g} N_{\beta_{\hat{s}_p}}} - (\frac{1}{\epsilon_p})^{\breve{g} I_{\beta_{\hat{s}_p}}})}); \epsilon_p \geq 1. \end{array} \right\} \end{aligned} \tag{14}$$

and $\beta_{\hat{s}_p}$ *are the exponential weights of* $\epsilon_p, (p = 1, 2, ..., n)$.

Proof. We demonstrate the aforementioned AO, $FOFWEA(\beta_{\hat{s}_1}, \beta_{\hat{s}_1}, ..., \beta_{\hat{s}_1})$ based on induction in mathematics n. Let $\epsilon_p \in (0, 1)$. Since $\beta_{\hat{s}_p}$ is FOFN for each p, $0 \leq P_{\beta_{\hat{s}_p}}, N_{\beta_{\hat{s}_p}}, I_{\beta_{\hat{s}_p}} \leq 1$ and $P_{\beta_{\hat{s}_p}}^{\breve{g}} + N_{\beta_{\hat{s}_p}}^{\breve{g}} + I_{\beta_{\hat{s}_p}}^{\breve{g}} \leq 1$.

Step 1: When $n = 2$, we can see that

$$\epsilon_1^{\beta_{\hat{s}_1}} = (\epsilon_1^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_1}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \epsilon_1^{\breve{g} N_{\beta_{\hat{s}_1}}}}, \sqrt[\breve{g}]{1 - \epsilon_1^{\breve{g} I_{\beta_{\hat{s}_1}}}}),$$

$$\epsilon_2^{\beta_{\hat{s}_2}} = (\epsilon_2^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_2}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \epsilon_2^{\breve{g} N_{\beta_{\hat{s}_2}}}}, \sqrt[\breve{g}]{1 - \epsilon_2^{\breve{g} I_{\beta_{\hat{s}_2}}}}).$$

are FOFNs. Then,

$$\begin{aligned} & FOFWEA(\beta_{\hat{s}_1}, \beta_{\hat{s}_2}) = \epsilon_1^{\beta_{\hat{s}_1}} \otimes \epsilon_2^{\beta_{\hat{s}_2}} \\ = & \left\{ \begin{array}{c} \epsilon_1^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_1}}^{\breve{g}})}} \epsilon_2^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_2}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \epsilon_1^{\breve{g} I_{\beta_{s1}}} \epsilon_2^{\breve{g} I_{\beta_{\hat{s}_2}}}}, \\ \sqrt[\breve{g}]{\epsilon_1^{\breve{g} N_{\beta_{\hat{s}_1}}} \epsilon_2^{\breve{g} N_{\beta_{\hat{s}_2}}} - (1 - \epsilon_1^{\breve{g} N_{\beta_{\hat{s}_1}}} - \epsilon_1^{\breve{g} I_{\beta_{\hat{s}_1}}})(1 - \epsilon_2^{\breve{g} N_{\beta_{\hat{s}_2}}} - \epsilon_2^{\breve{g} I_{\beta_{\hat{s}_2}}})}) \end{array} \right\} \end{aligned}$$

$$= \left\{ \Pi_{p=1}^2 \epsilon_p^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_p}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \Pi_{p=1}^2 \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}}}, \sqrt[\breve{g}]{\Pi_{p=1}^2 \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \Pi_{p=1}^2 (1 - \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \epsilon_p^{\breve{g} I_{\beta_{\hat{s}_p}}})}) \right\}$$

is also an FOFN in the extended interval $[1, 2]$.

Step 2: Consider that the aggregation operator $FOFWEA(\beta_{\hat{s}_1}, \beta_{\hat{s}_2}, ..., \beta_{\hat{s}_n})$ holds for $n = K$. Then,

$$FOFWEA(\beta_{\hat{s}_1}, \beta_{\hat{s}_2}, ..., \beta_{\hat{s}_K}) = \left\{ (\Pi_{p=1}^{K} \epsilon_p^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_p}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \Pi_{p=1}^{2} \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}}}, \sqrt[\breve{g}]{\Pi_{p=1}^{2} \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \Pi_{p=1}^{2} (1 - \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \epsilon_p^{\breve{g} I_{\beta_{\hat{s}_p}}})} \right\}$$

and the aggregated value is an FOFN.

Step 3: When $n = K + 1$, we have

$$FOFWEA(\beta_{\hat{s}_1}, \beta_{\hat{s}_2}, ..., \beta_{\hat{s}_K}) = \epsilon_1^{\beta_{\hat{s}_1}} \otimes \epsilon_2^{\beta_{\hat{s}_2}} \otimes ... \otimes \epsilon_K^{\beta_{\hat{s}_K}} \otimes \epsilon_{K+1}^{\beta_{\hat{s}_{K+1}}}$$

$$= \left\{ \begin{array}{c} \left(\begin{array}{c} \Pi_{p=1}^{K} \epsilon_p^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_p}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \Pi_{p=1}^{K} \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}}}, \\ \sqrt[\breve{g}]{\Pi_{p=1}^{K} \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \Pi_{p=1}^{K} (1 - \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \epsilon_p^{\breve{g} I_{\beta_{\hat{s}_p}}})} \end{array} \right) \\ \otimes \left(\epsilon_{K+1}^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_{K+1}}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \epsilon_{K+1}^{\breve{g} N_{\beta_{\hat{s}_{K+1}}}}}, \sqrt[\breve{g}]{\epsilon_{K+1}^{\breve{g} N_{\beta_{\hat{s}_{K+1}}}} - (1 - \epsilon_{K+1}^{\breve{g} N_{\beta_{\hat{s}_{K+1}}}} - \epsilon_{K+1}^{\breve{g} I_{\beta_{\hat{s}_{K+1}}}})} \right) \end{array} \right\}$$

$$= \left(\Pi_{p=1}^{K+1} \epsilon_p^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_p}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \Pi_{p=1}^{K+1} \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}}}, \sqrt[\breve{g}]{\Pi_{p=1}^{K+1} \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \Pi_{p=1}^{K+1} (1 - \epsilon_p^{\breve{g} N_{\beta_{\hat{s}_p}}} - \epsilon_p^{\breve{g} I_{\beta_{\hat{s}_p}}})} \right),$$

whose aggregated value is also an FOFN in the extended interval $[1,2]$. Therefore, Definition 13 holds. On the other hand, when $\epsilon_p \geq 1$, and $0 \leq \frac{1}{\epsilon_p} \leq 1$, we can also obtain

$$FOFWEA(\beta_{\hat{s}_1}, \beta_{\hat{s}_2}, ..., \beta_{\hat{s}_K}) = \left\{ \begin{array}{c} \Pi_{p=1}^{n} (\frac{1}{\epsilon_p})^{\sqrt[\breve{g}]{1-(P_{\beta_{\hat{s}_p}}^{\breve{g}})}}, \sqrt[\breve{g}]{1 - \Pi_{p=1}^{n} (\frac{1}{\epsilon_p})^{\breve{g} N_{\beta_{\hat{s}_p}}}}, \\ \sqrt[\breve{g}]{\Pi_{p=1}^{n} (\frac{1}{\epsilon_p})^{\breve{g} N_{\beta_{\hat{s}_p}}} - \Pi_{p=1}^{n} (1 - (\frac{1}{\epsilon_p})^{\breve{g} N_{\beta_{\hat{s}_p}}} - (\frac{1}{\epsilon_p})^{\breve{g} I_{\beta_{\hat{s}_p}}}} \end{array} \right\}$$

and the aggregated value is an FOFN in the prolonged range $[1,2]$. Hence, the proof. □

4. Einstein Exponential Operational Laws of FOFSs

This section will discuss the Einstein exponential operational laws of fractions for the FOFS model, as well as few of its features.

Definition 9. *Let* $\epsilon^{\breve{A}_{\hat{s}}} = (\epsilon^{\sqrt[\breve{g}]{1-P_{\breve{A}_{\hat{s}}}^{\breve{g}}}}, \sqrt[\breve{g}]{1 - \epsilon^{\breve{g} N_{\breve{A}_{\hat{s}}}}}, \sqrt[\breve{g}]{1 - \epsilon^{\breve{g} I_{\breve{A}_{\hat{s}}}}}$ *and* $\epsilon^{B_{\hat{s}}} = (\epsilon^{\sqrt[\breve{g}]{1-P_{B_{\hat{s}}}^{\breve{g}}}}, \sqrt[\breve{g}]{1 - \epsilon^{\breve{g} N_{B_{\hat{s}}}}}, \sqrt[\breve{g}]{1 - \epsilon^{\breve{g} I_{B_{\hat{s}}}}}$ *be two families of exponential fractional orthotriple fuzzy numbers, for all* $\epsilon \in (0,1)$*, where* $\breve{A}_{\hat{s}} = (P_{\breve{A}_{\hat{s}}}, N_{\breve{A}_{\hat{s}}}, I_{\breve{A}_{\hat{s}}})$*,* $B_{\hat{s}} = (P_{B_{\hat{s}}}, N_{B_{\hat{s}}}, I_{\breve{A}_{\hat{s}}})$ *are fractional orthotriple fuzzy numbers. Then, the fundamental Einstein EO principles of fractional orthotriple fuzzy operators are as mentioned, following:*

1. $$\epsilon^{\breve{A}_{\hat{s}}} \oplus \epsilon^{B_{\hat{s}}} = \left\{ \left\langle \begin{array}{c} \frac{\epsilon^{\sqrt[\breve{g}]{1-P_{\breve{A}_{\hat{s}}}^{\breve{g}}}} + \epsilon^{\sqrt[\breve{g}]{1-P_{B_{\hat{s}}}^{\breve{g}}}}}{1 + \left(\epsilon^{\sqrt[\breve{g}]{1-P_{\breve{A}_{\hat{s}}}^{\breve{g}}}}\right) \cdot \left(\epsilon^{\sqrt[\breve{g}]{1-P_{B_{\hat{s}}}^{\breve{g}}}}\right)}, \frac{\sqrt[\breve{g}]{1-\epsilon^{\breve{g} N_{\breve{A}_{\hat{s}}}}} \cdot \sqrt[\breve{g}]{1-\epsilon^{\breve{g} N_{B_{\hat{s}}}}}}{1 + \left(1 - \sqrt[\breve{g}]{1-\epsilon^{\breve{g} N_{\breve{A}_{\hat{s}}}}}\right) \cdot \left(1 - \sqrt[\breve{g}]{1-\epsilon^{\breve{g} N_{B_{\hat{s}}}}}\right)}, \\ \frac{\sqrt[\breve{g}]{1-\epsilon^{\breve{g} I_{\breve{A}_{\hat{s}}}}} \cdot \sqrt[\breve{g}]{1-\epsilon^{\breve{g} I_{B_{\hat{s}}}}}}{1 + \left(1 - \sqrt[\breve{g}]{1-\epsilon^{\breve{g} I_{\breve{A}_{\hat{s}}}}}\right) \cdot \left(1 - \sqrt[\breve{g}]{1-\epsilon^{\breve{g} I_{B_{\hat{s}}}}}\right)} \end{array} \right\rangle \right\};$$

$$2. \quad \epsilon^{\breve{A}_{\hat{s}}} \otimes \epsilon^{B_{\hat{s}}} = \left\{ \left\langle \begin{array}{c} \dfrac{\left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\breve{A}_{\hat{s}}}}}\right).\left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{B_{\hat{s}}}}}\right)}{1+\left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\breve{A}_{\hat{s}}}}}\right).\left(\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{B_{\hat{s}}}}}\right)}, \dfrac{\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\breve{A}_{\hat{s}}}}}+\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{B_{\hat{s}}}}}}{1+\left(1-\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\breve{A}_{\hat{s}}}}}\right).\left(1-\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{B_{\hat{s}}}}}\right)}, \\ \dfrac{\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\breve{A}_{\hat{s}}}}}+\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{B_{\hat{s}}}}}}{1+\left(1-\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\breve{A}_{\hat{s}}}}}\right).\left(1-\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{B_{\hat{s}}}}}\right)} \end{array} \right\rangle \right\};$$

$$3. \quad K.\varepsilon(\epsilon^{\breve{A}_{\hat{s}}}) = \left\{ \left\langle \begin{array}{c} \dfrac{\left[1+\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\breve{A}_{\hat{s}}}}}\right]^K - \left[1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\breve{A}_{\hat{s}}}}}\right]^K}{\left[1+\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\breve{A}_{\hat{s}}}}}\right]^K + \left[1-\epsilon^{\sqrt[\breve{g}]{1-P^{\breve{g}}_{\breve{A}_{\hat{s}}}}}\right]^K}, \dfrac{2\left[\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\breve{A}_{\hat{s}}}}}\right]^K}{\left[2-\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\breve{A}_{\hat{s}}}}}\right]^K+\left[\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\breve{A}_{\hat{s}}}}}\right]^K}, \\ \dfrac{2\left[\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\breve{A}_{\hat{s}}}}}\right]^K}{\left[2-\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\breve{A}_{\hat{s}}}}}\right]^K+\left[\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\breve{A}_{\hat{s}}}}}\right]^K} \end{array} \right\rangle \right\}.$$

Theorem 7. *Let $\epsilon^{\beta_{\hat{s}}}, \epsilon^{\beta_{\hat{s}_1}}$, and $\epsilon^{\beta_{\hat{s}_2}}$ be three-exponent family fractional orthotriple fuzzy numbers of $\beta_{\hat{s}} = (P_{\beta_{\hat{s}}}, N_{\beta_{\hat{s}}}, I_{\beta_{\hat{s}}}), \beta_{\hat{s}_1} = (P_{\beta_{\hat{s}_1}}, N_{\beta_{\hat{s}_1}}, I_{\beta_{\hat{s}_1}})$ and $\beta_{\hat{s}_2} = (P_{\beta_{\hat{s}_2}}, N_{\beta_{\hat{s}_2}}, I_{\beta_{\hat{s}_2}})$; correspondingly, $K_1, K_2, K_3 > 0$ be three real numbers, and $\epsilon \in (0,1)$. Then, there are a few more:*

(1) $\epsilon^{\beta_{\hat{s}_1}} \oplus \varepsilon\lambda^{\beta_{\hat{s}_2}} = \epsilon^{\beta_{s2}} \oplus \varepsilon\lambda^{\beta_{\hat{s}_1}}$;
(2) $K.\varepsilon(\epsilon^{\beta_{\hat{s}_1}} \oplus \varepsilon\lambda^{\beta_{\hat{s}_2}}) = K.\varepsilon(\epsilon^{\beta_{\hat{s}_1}}) \oplus \varepsilon K.\varepsilon(\epsilon^{\beta_{\hat{s}_2}})$;
(3) $K_1.\varepsilon(\epsilon^{\beta_{\hat{s}}}) \oplus \varepsilon K_2.\varepsilon(\epsilon^{\beta_{\hat{s}}}) = (K_1 + K_2).\varepsilon(\epsilon^{\beta_{\hat{s}}})$;
(4) $(K_1.K_2).\varepsilon^{\epsilon^{\beta_{\hat{s}}}} = K_1.\varepsilon(K_2.\varepsilon^{\epsilon^{\beta_{\hat{s}}}})$.

Proof. The proof is comparable to the proof of Theorem 4, and, as a result, is left out. □

Einstein Exponential Aggregation Operator for FOFNs

This section introduces fractional orthotriple fuzzy, a novel Einstein exponential aggregation operator utilizing fractional orthotriple fuzzy information. To aggregate fractional orthotriple fuzzy information, the Einstein weighted exponential averaging operator was created.

Definition 10. *Let $\epsilon_p^{\beta_{\hat{s}_p}}, (p = 1, 2, ..., n)$ be a family of exponential of FOFNs with respect to $\beta_{\hat{s}_p} = (P_{\beta_p}, N_{\beta_p}, I_{\beta_p})$, where ϵ_p are real values. Let $K_p = (K_1, K_2, ..., K_n)^T$ be the weighting vector of $\epsilon^{\beta_{\hat{s}_p}}$ $(p = 1, 2, ..., n)$, in such a way that $K_p \in [0,1]$ $\sum_{p=1}^n K_p = 1$; then, for an FOFEWEA, the operation on n is a mapping. $FOFEWEA : (\epsilon^{\beta_{\hat{s}}})^* \to \epsilon^{\beta_{\hat{s}}}$, and*

$$FOFEWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, ..., \epsilon_n^{\beta_{\hat{s}_n}}) = K_1.\varepsilon\epsilon_1^{\beta_{\hat{s}_1}} \oplus \varepsilon K_2.\varepsilon\epsilon_2^{\beta_{\hat{s}_2}} \oplus \varepsilon ... \oplus \varepsilon K_1.\varepsilon\epsilon_n^{\beta_{\hat{s}_n}} \quad (15)$$

Theorem 8. *Let $\epsilon_p^{\beta_{\hat{s}_p}}, (p = 1, 2, ..., n)$ be an exponential family of FOFNs with respect to $\beta_{\hat{s}_p} = (P_{\beta_p}, N_{\beta_p}, I_{\beta_p})$; then, using the aggregated numbers, the FOFEWEA operator is also a type of FOFN in the extended interval $[1, 2]$, and*

$$FOFWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, \ldots, \epsilon_n^{\beta_{\hat{s}_n}}) \tag{16}$$
$$= \left\{ \begin{array}{c} \left\langle \dfrac{\Pi_{p=1}^{n}\left(1+\epsilon^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p} - \Pi_{p=1}^{n}\left(1-\epsilon^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p}}{\Pi_{p=1}^{n}\left(1+\epsilon^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p} + \Pi_{p=1}^{n}\left(1-\epsilon^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p}} \right. \\ \dfrac{2\Pi_{p=1}^{n}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p}}{\Pi_{p=1}^{n}\left(2-\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p} + \Pi_{p=1}^{n}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p}} \\ \left. \dfrac{2\Pi_{p=1}^{n}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p}}{\Pi_{p=1}^{n}\left(2-\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p} + \Pi_{p=1}^{n}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p}} \right\rangle ; \epsilon_p \in (0,1). \end{array} \right\}.$$

where $K_p = (K_1, K_2, \ldots, K_n)^T$ *is the weighting vector of* $\beta_{\hat{s}_p} = (P_{\beta_p}, N_{\beta_p}, I_{\beta_p})$ *in such a way that* $K_p \in [0,1], (p = 1, 2, \ldots, n)$ *and* $\sum_{p=1}^{n} K_p = 1$.

Proof. We will use the mathematical induction principle to prove this theorem. It is evident that Equation (16) is true for $n = 1$. Now, suppose that Equation (16) is true for $n = \delta$, i.e.,

$$FOFWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, \ldots, \epsilon_\delta^{\beta_{\hat{s}_\delta}})$$
$$= \left\{ \begin{array}{c} \left\langle \dfrac{\Pi_{p=1}^{\delta}\left(1+\epsilon^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p} - \Pi_{p=1}^{\delta}\left(1-\epsilon^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p}}{\Pi_{p=1}^{\delta}\left(1+\epsilon^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p} + \Pi_{p=1}^{\delta}\left(1-\epsilon^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p}} \right. \\ \dfrac{2\Pi_{p=1}^{\delta}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p}}{\Pi_{p=1}^{\delta}\left(2-\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p} + \Pi_{p=1}^{\delta}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p}} \\ \left. \dfrac{2\Pi_{p=1}^{\delta}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p}}{\Pi_{p=1}^{\delta}\left(2-\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p} + \Pi_{p=1}^{\delta}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p}} \right\rangle ; \epsilon_p \in (0,1). \end{array} \right\}$$

Let

$$t_1 = \Pi_{p=1}^{\delta}\left(1+\epsilon_p^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p}, w_1 = \Pi_{p=1}^{\delta}\left(1-\epsilon_p^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p},$$

$$q_1 = \Pi_{p=1}^{\delta}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p}, r_1 = \Pi_{p=1}^{\delta}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p},$$

$$e_1 = \Pi_{p=1}^{\delta}\left(2-\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p}, f_1 = \Pi_{p=1}^{\delta}\left(2-\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p}$$

$$FOFWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, ..., \epsilon_n^{\beta_{\hat{s}_n}}) = \left\langle \frac{t_1 - w_1}{t_1 + w_1}, \frac{2q_1}{e_1 + q_1}, \frac{2r_1}{f_1 + r_1} \right\rangle$$

Then, if $n = \delta + 1$, we have

$$\begin{aligned} & FOFWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, ..., \epsilon_{\delta+1}^{\beta_{\hat{s}_{\delta+1}}}) \\ = \; & K_1.\varepsilon(\epsilon_1^{\beta_{\hat{s}_1}}) \oplus \varepsilon ... \oplus K_\delta.\varepsilon(\epsilon_\delta^{\beta_{\hat{s}_\delta}}) \oplus \varepsilon K_{\delta+1}.\varepsilon(\epsilon_{\delta+1}^{\beta_{\hat{s}_{\delta+1}}}) \\ = \; & FOFWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, ..., \epsilon_\delta^{\beta_{\hat{s}_\delta}}) \oplus \varepsilon K_{\delta+1}.\varepsilon(\epsilon_{\delta+1}^{\beta_{\hat{s}_{\delta+1}}}) \end{aligned}$$

Let

$$t_2 = \left(1 + \epsilon_{\delta+1}^{\sqrt[\breve{g}]{1 - P_{\beta_{\hat{s}_{\delta+1}}}^{\breve{g}}}}\right)^{K_{\delta+1}}, w_2 = \left(1 - \epsilon_{\delta+1}^{\sqrt[\breve{g}]{1 - P_{\beta_{\hat{s}_{\delta+1}}}^{\breve{g}}}}\right)^{K_{\delta+1}},$$

$$q_2 = \left(\sqrt[\breve{g}]{1 - \epsilon_{\delta+1}^{\breve{g}N_{\beta_{\hat{s}_{\delta+1}}}}}\right)^{K_{\delta+1}}, r_2 = \left(\sqrt[\breve{g}]{1 - \epsilon_{\delta+1}^{\breve{g}I_{\beta_{\hat{s}_{\delta+1}}}}}\right)^{K_{\delta+1}}$$

$$e_2 = \left(2 - \sqrt[\breve{g}]{1 - \epsilon_{\delta+1}^{\breve{g}N_{\beta_{\hat{s}_{\delta+1}}}}}\right)^{K_{\delta+1}}, f_2 = \left(2 - \sqrt[\breve{g}]{1 - \epsilon_{\delta+1}^{\breve{g}I_{\beta_{\hat{s}_{\delta+1}}}}}\right)^{K_{\delta+1}}$$

Then,

$$K_{\delta+1}.\varepsilon(\epsilon_{\delta+1}^{\beta_{\hat{s}_{\delta+1}}}) = \left\langle \frac{t_2 - w_2}{t_2 + w_2}, \frac{2q_2}{e_2 + c_2}, \frac{2r_2}{f_2 + d_2} \right\rangle;$$

As a result of Einstein's operational law, we have

$$\begin{aligned} & FOFEWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, ..., \epsilon_{\delta+1}^{\beta_{\hat{s}_{\delta+1}}}) \\ = \; & FOFWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, ..., \epsilon_\delta^{\beta_{\hat{s}_\delta}}) \oplus \varepsilon K_{\delta+1}.\varepsilon(\epsilon_{\delta+1}^{\beta_{\hat{s}_{\delta+1}}}) \end{aligned}$$

$$\begin{aligned} = \; & \left\langle \frac{t_1 - w_1}{t_1 + w_1}, \frac{2q_1}{e_1 + q_1}, \frac{2r_1}{f_1 + r_1} \right\rangle \oplus \varepsilon \left\langle \frac{t_2 - w_2}{t_2 + w_2}, \frac{2q_2}{e_2 + q_2}, \frac{2r_2}{f_2 + r_2} \right\rangle \\ = \; & \left\langle \frac{t_1 t_2 - w_1 w_2}{t_1 t_2 + w_1 w_2}, \frac{2q_1 q_2}{e_1 e_2 + q_1 q_2}, \frac{2r_1 r_2}{f_1 f_2 + r_1 r_2} \right\rangle \end{aligned}$$

$$\begin{aligned} & FOFWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, ..., \epsilon_{\delta+1}^{\beta_{\hat{s}_{\delta+1}}}) \\ = & \left\{ \begin{array}{c} \dfrac{\Pi_{p=1}^{\delta+1}\left(1+\epsilon_p^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p} - \Pi_{p=1}^{\delta+1}\left(1-\epsilon_p^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p}}{\Pi_{p=1}^{\delta+1}\left(1+\epsilon_p^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p} + \Pi_{p=1}^{\delta+1}\left(1-\epsilon_p^{\sqrt[\breve{g}]{1-P_{\beta_{\hat{s}_p}}^{\breve{g}}}}\right)^{K_p}}, \\ \dfrac{2\Pi_{p=1}^{\delta}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p}}{\Pi_{p=1}^{\delta+1}\left(2-\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p} + \Pi_{p=1}^{\delta+1}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}N_{\beta_{\hat{s}_p}}}}\right)^{K_p}}, \\ \dfrac{2\Pi_{p=1}^{\delta+1}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p}}{\Pi_{p=1}^{\delta+1}\left(2-\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p} + \Pi_{p=1}^{\delta+1}\left(\sqrt[\breve{g}]{1-\epsilon_p^{\breve{g}I_{\beta_{\hat{s}_p}}}}\right)^{K_p}} \end{array} ; \epsilon_p \epsilon (0,1) \right\} \end{aligned}$$

Hence, (16) is true for $n = \delta + 1$. Therefore, (16) is true for all n, and this completes the theorem's proof. □

Similarly, we can easily obtain the $FOFEWEA(\epsilon_1^{\beta_{\hat{s}_1}}, \epsilon_2^{\beta_{\hat{s}_2}}, ..., \epsilon_{\delta+1}^{\beta_{\hat{s}_{\delta+1}}})$ operator, the value of $\epsilon_p \geq 1$, and $0 \leq \frac{1}{\epsilon_p} \leq 1$. Also, the aggregated figures are FOFN in the extended interval $[1, 2]$.

5. Decision-Making Algorithm Based on the Proposed Aggregation Operators

This section suggests an MCGDM technique dependent upon operators, which includes the phases listed below:

Step 1: Consider a decision-making problem with n choices $\breve{A}_p(p = 1, 2, ..., n)$ and m attributes $B_p(j = 1, 2, ..., m)$ whose fractional orthotriple fuzzy weight vector values are $\beta_{\hat{s}_j} = (\beta_{\hat{s}_1}, \beta_{\hat{s}_2}, ..., \beta_{\hat{s}_m})(j = 1, 2, ..., m)$ in such a way that $\beta_{\hat{s}_j} = (P_{\beta_{\hat{s}_j}}, N_{\beta_{\hat{s}_j}}, I_{\beta_{\hat{s}_j}})$; $0 \leq P_{\beta_{\hat{s}_j}}, N_{\beta_{\hat{s}_j}}, I_{\beta_{\hat{s}_j}} \leq 1$ and $0 \leq P^{\breve{g}}_{\beta_{\hat{s}_j}} + N^{\breve{g}}_{\beta_{\hat{s}_j}} + I^{\breve{g}}_{\beta_{\hat{s}_j}} \leq 1$. The offered choices are then reviewed by experts based on a set of qualities, and they provide their preferred numbers based on the fuzzy data signified by $\epsilon_{ij}(p = 1, 2, ..., n)(j = 1, 2, ..., m)$ and $0 \leq \epsilon_{ij} \leq 1$. In general, there are two kinds of attributes: the first one being the benefit type (B_1) and another cost type (B_2). If the MCGDM characteristics are of a similar kind, the preferred numbers are not necessary to normalize. If the characteristics are of different types, we use the formula below to convert the benefit type of preferred numbers to cost type values.

$$\epsilon_{ij} = \begin{cases} \epsilon_{ij}; j \in B_1 \\ \epsilon_{ij}^c; j \in B_2 \end{cases}$$

Step 2: Use the operators of aggregation in such a way that FOFWEA, FOFOWEA, FOFHWEA, FOFEWEA, FOFEOWEA, and FOFEHWEA will aggregate each different alternative's preference ratings into aggregate values $\alpha_p(p = 1, 2, ..., n)$.

Step 3: Compute the aggregated score numbers FOFNs $\alpha_p(p = 1, 2, ..., n)$.

Step 4: By using exponential aggregation operators, we compute the aggregated values.

Step 5: Ranking the choices depends upon their score numbers.

In Section 6, the aforementioned technique will be explained using a real-world numerical example.

6. Illustrative Example

In this part, we use the suggested approach to solve a mathematical issue related to the improvement of an online bank security management system in order to illustrate its significance and reliability.

Description of the Problem

As the world becomes increasingly automated, Internet-based banking has become a fundamental component of our everyday life. Yet, the rising popularity of internet-based banking has culminated in an increase in haphazardness and fraud. There have been numerous instances of cyberattacks on banking organizations in the past decade, leading to enormous amounts of money lost and loss of client trust. To solve this problem, five suggestions $\breve{A}_1, \breve{A}_2, \breve{A}_3, \breve{A}_4$, and $\breve{A}_5$ have been provided by a security firm. A popular bank in Pakistan intends to pick the best possible option for banking security and has appointed three safety specialists D_1, D_2, and D_3 that serve as a specialist with weight vector $w = (w_1, w_2, w_3) = (0.35, 0.40, 0.25)^T$, which demonstrates the experts' respective skill. The panel of specialists identified five essential requirements after determining the security elements and attacks on computers on online banking platforms, Z_1, Z_2, Z_3, Z_4, and Z_5, upon which the best suggestion is chosen. The significance of these criteria are described below:

Z_1: **Compliance**

The safety system should adhere to the suitable legal requirements and monetary safety regulations, such as PCI DSS, ISO 27001, and NIST. As a result, conformance is critical in Internet banking to secure user data, preserve fairness, minimize risk, promote confidence, and prevent legal problems. Banks may establish a secure and safe atmosphere for users to conduct banking operations by adhering to legislation and applying efficient safety precautions.

Z_2 : **Encryption**

Encryption is a crucial safety component of internet banking because it secures confidential client information from illicit access and aids in data breach detection. Encrypting user data permits banks to enhance trust among customers while additionally minimizing their accountability in the event of a breach of security.

Z_3 : **Multifactor Authentication**

MFA (Multifactor Authentication) plays a vital role in internet banking security because it brings another level safety to user accounts. MFA demands that users give at least two kinds of identification, such as their username and password and a security token, before they may access their account information. This makes it far more difficult for cybercriminals to obtain illicit entry to the account, although they have stolen the login information.

Z_4 : **Access Control, Incident Response, and Recovery**

Banking security rests substantially on controls over access, response to emergencies, and recovery. By taking these actions, banks may help to secure sensitive money-related information, limit the impact on safety events, and recover quickly in the event of an incident of security.

Z_5 : **Employee Training and Awareness**

Bankers and financial organizations should teach their personnel best practices in cybersecurity, such as recognizing and reacting to security hazards. Here are some particular reasons why awareness and education are vital for banking employees: they minimize human error, raise awareness, improve reaction times, and foster a secure environment.

Step 1. We obtanied FOF information given in Tables 1–3.

Table 1. Opportunities investing in a wealth management organization R^1.

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(0.7, 0.6, 0.2)$	$(0.4, 0.8, 0.5)$	$(0.2, 0.9, 0.6)$	$(0.3, 0.7, 0.5)$
Z_2	$(0.1, 0.4, 0.8)$	$(0.5, 0.7, 0.3)$	$(0.5, 0.6, 0.6)$	$(0.3, 0.5, 0.2)$
Z_3	$(0.3, 0.6, 0.3)$	$(0.7, 0.8, 0.1)$	$(0.2, 0.6, 0.8)$	$(0.6, 0.2, 0.7)$
Z_4	$(0.2, 0.7, 0.6)$	$(0.2, 0.6, 0.7)$	$(0.3, 0.7, 0.8)$	$(0.4, 0.3, 0.7)$
Z_5	$(0.3, 0.5, 0.7)$	$(0.2, 0.4, 0.7)$	$(0.6, 0.9, 0.2)$	$(0.5, 0.7, 0.2)$

Table 2. Opportunities investing in a wealth management organization R^2.

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(0.8, 0.6, 0.4)$	$(0.7, 0.6, 0.4)$	$(0.5, 0.2, 0.3)$	$(0.4, 0.3, 0.5)$
Z_2	$(0.3, 0.8, 0.6)$	$(0.5, 0.6, 0.8)$	$(0.4, 0.1, 0.7)$	$(0.6, 0.3, 0.6)$
Z_3	$(0.4, 0.6, 0.7)$	$(0.2, 0.4, 0.4)$	$(0.6, 0.3, 0.6)$	$(0.7, 0.3, 0.1)$
Z_4	$(0.2, 0.6, 0.8)$	$(0.6, 0.7, 0.5)$	$(0.7, 0.3, 0.5)$	$(0.6, 0.3, 0.6)$
Z_5	$(0.4, 0.5, 0.7)$	$(0.4, 0.6, 0.8)$	$(0.5, 0.8, 0.2)$	$(0.4, 0.6, 0.8)$

Table 3. Opportunities investing in a wealth management organization R^3.

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(0.2, 0.7, 0.4)$	$(0.2, 0.4, 0.7)$	$(0.2, 0.6, 0.7)$	$(0.4, 0.5, 0.8)$
Z_2	$(0.7, 0.4, 0.5)$	$(0.8, 0.1, 0.4)$	$(0.3, 0.5, 0.7)$	$(0.3, 0.9, 0.5)$
Z_3	$(0.6, 0.7, 0.3)$	$(0.5, 0.7, 0.2)$	$(0.3, 0.6, 0.3)$	$(0.4, 0.6, 0.2)$
Z_4	$(0.3, 0.6, 0.7)$	$(0.4, 0.6, 0.2)$	$(0.2, 0.3, 0.7)$	$(0.4, 0.2, 0.5)$
Z_5	$(0.4, 0.6, 0.2)$	$(0.5, 0.3, 0.6)$	$(0.3, 0.7, 0.2)$	$(0.5, 0.3, 0.6)$

Step 2. We use FOFWEA on Tables 1–3 to obtain the following results given in Table 4.

Table 4. Aggregated values using FOFWEA operator.

	$\breve{A}_1$	$\breve{A}_2$
Z_1	$(0.8682, 0.7992, 0.9186)$	$(0.8428, 0.7832, 0.8724)$
Z_2	$(0.8404, 0.7832, 0.8808)$	$(0.8624, 0.8269, 0.8196)$
Z_3	$(0.8368, 0.8389, 0.8533)$	$(0.8462, 0.8602, 0.8271)$
Z_4	$(0.8223, 0.8389, 0.8495)$	$(0.8358, 0.8269, 0.8322)$
Z_5	$(0.8284, 0.7832, 0.9183)$	$(0.8308, 0.7236, 0.9313)$
	$\breve{A}_3$	$\breve{A}_4$
	$(0.8277, 0.8602, 0.7591)$	$(0.8284, 0.8389, 0.8123)$
	$(0.8318, 0.7992, 0.8368)$	$(0.8348, 0.8269, 0.8358)$
	$(0.8337, 0.7832, 0.8841)$	$(0.8545, 0.7832, 0.9164)$
	$(0.8408, 0.7655, 0.9028)$	$(0.8389, 0.7236, 0.9239)$
	$(0.8402, 0.8862, 0.7920)$	$(0.8372, 0.8501, 0.8425)$

Step 3. By using exponential aggregation operators, we computed the aggregated values, which are given in Tables 5–7.

Table 5. Aggregated value using FOFWEA operator.

Z_1	$(0.5937, 0.9503, 0.4739)$
Z_2	$(0.5944, 0.9399, 0.4608)$
Z_3	$(0.5965, 0.9419, 0.4716)$
Z_4	$(0.5899, 0.9388, 0.4977)$
Z_5	$(0.5895, 0.9349, 0.4812)$

Table 6. Aggregated value using FOFOWEA operator.

Z_1	$(0.6035, 0.9521, 0.5128)$
Z_2	$(0.5931, 0.9412, 0.5227)$
Z_3	$(0.5929, 0.9413, 0.5224)$
Z_4	$(0.5912, 0.9369, 0.5341)$
Z_5	$(0.5871, 0.9402, 0.5254)$

Table 7. Aggregated value using FOFHWEA operator.

Z_1	$(0.2563, 0.9702, 0.4921)$
Z_2	$(0.2039, 0.9484, 0.4826)$
Z_3	$(0.2004, 0.9507, 0.4887)$
Z_4	$(0.1754, 0.9476, 0.5121)$
Z_5	$(0.1761, 0.9431, 0.4964)$

Step 4. By using Einstein exponentonal aggregation operators, we computed the aggregated values, which are given in Tables 8–10.

Table 8. Aggregated value using FOFEWEA operator.

Z_1	$(0.8632, 0.6794, 0.5228)$
Z_2	$(0.8644, 0.6779, 0.4904)$
Z_3	$(0.8639, 0.6787, 0.5095)$
Z_4	$(0.8623, 0.6716, 0.5429)$
Z_5	$(0.8620, 0.6760, 0.5135)$

Table 9. Aggregated value using FOFEOWEA operator.

Z_1	$(0.8651, 0.6829, 0.6842)$
Z_2	$(0.8627, 0.6772, 0.6881)$
Z_3	$(0.8632, 0.6776, 0.6892)$
Z_4	$(0.8624, 0.6702, 0.6902)$
Z_5	$(0.8611, 0.6754, 0.6869)$

Table 10. Aggregated value using FOFEHWEA operator.

Z_1	$(0.8566, 0.6658, 0.4719)$
Z_2	$(0.8561, 0.6655, 0.4347)$
Z_3	$(0.8559, 0.6664, 0.4421)$
Z_4	$(0.8543, 0.6572, 0.4799)$
Z_5	$(0.8541, 0.6594, 0.4583)$

Step 5. The score value and ranking of the alternatives of all operators is given in Table 11.

Table 11. Score value and ranking using defined aggregation operators.

Operator	$\hat{s}(Z_1)$	$\hat{s}(Z_2)$	$\hat{s}(Z_3)$	$\hat{s}(Z_4)$	$\hat{s}(Z_5)$	**Ranking**
$FOFWEA$	-0.212	-0.211	-0.205	-0.186	-0.194	$Z_4 > Z_5 > Z_3 > Z_1 > Z_2$
$FOFOWEA$	-0.184	-0.170	-0.170	-0.158	-0.168	$Z_4 > Z_5 > Z_2 > Z_3 > Z_1$
$FOFHWEA$	-0.172	-0.139	-0.130	-0.076	-0.096	$Z_4 > Z_5 > Z_3 > Z_2 > Z_1$
$FOFEWEA$	0.091	0.104	0.096	0.085	0.095	$Z_4 > Z_1 > Z_5 > Z_3 > Z_2$
$FOFEOWEA$	0.032	0.030	0.030	0.029	0.030	$Z_4 > Z_3 > Z_5 > Z_2 > Z_1$
$FOFEHWEA$	0.110	0.124	0.120	0.108	0.116	$Z_4 > Z_1 > Z_5 > Z_3 > Z_2$

7. Comparison Analysis

In this section, we build the framework for a collaborative examination of the previously given material by employing a few well-known methodologies. The key points of the discussion are outlined below.

Here, we compare our work with the following concepts: Einstein exponential operation laws of spherical fuzzy sets and aggregation operators in decision making [44]; fractional orthotriple fuzzy rough Hamacher aggregation operators and their application on service quality of wireless network selection [43]; fractional orthotriple fuzzy Choquet–Frank aggregation operators and their application in optimal selection for EEG of depression patients [45]. The final results are presented in Table 12 for this discussion.

Table 12. Different aggregation operators and score values and ranking.

Operator	$\hat{s}(Z_1)$	$\hat{s}(Z_2)$	$\hat{s}(Z_3)$	$\hat{s}(Z_4)$	$\hat{s}(Z_5)$	**Ranking**
SFWEA [44]	-0.184	-0.170	-0.170	-0.158	-0.168	$Z_4 > Z_5 > Z_2 > Z_3 > Z_1$
SFEWEA [44]	-0.184	-0.170	-0.170	-0.158	-0.168	$Z_4 > Z_5 > Z_2 > Z_3 > Z_1$
FOFRHWA [43]	0.287	0.319	0.318	0.237	0.266	$Z_4 > Z_5 > Z_1 > Z_3 > Z_2$
FOFRHWG [43]	0.091	0.104	0.096	0.085	0.095	$Z_4 > Z_1 > Z_5 > Z_3 > Z_2$
FOFCFWA [45]	0.032	0.030	0.030	0.029	0.030	$Z_4 > Z_1 > Z_5 > Z_3 > Z_2$
FOFCFWA [45]	0.110	0.124	0.120	0.108	0.116	$Z_4 > Z_3 > Z_5 > Z_2 > Z_1$

Sensitivity Analysis

The suggested approach yielded the subsequent results of fractional orthotriple fuzzy operators of aggregation and the present famous fuzzy MCGDM framework that are presented in Table 11, and we find that the alternative Z_4 is suggested as the ideal solution.

From Table 11, we are able to observe that the order of rank determined by the FOFWEA operator differs significantly from the other approaches because of its specific characteristics. The FOFEWEA operator works with fractional fuzzy numbers additional to the assigned values. Therefore, when employing the FOFEWEA operator, the score numbers are positive; however, regardless of the weighing factors of the characteristics, the FOFEWEA operatortion generates negative grades. This examination considers the techniques' and suggested operators' reliability.

8. Conclusions

Fractional orthotriple sets are more strong tools than picture fuzzy sets and spherical fuzzy sets, and they provide extra options to decision-makers for several real-life problems. Also, aggregation operators want to reduce the set of finite values into one value, so they were motivated by the generality of fractional orthotriple fuzzy set and basic characteristics of aggregation operators during this article. In this paper, the exponential and Einstein exponential operational laws for the FOFS are defined, and their distinctive characteristics are thoroughly studied. The main finding in the exponential operational laws of FOFSs is that by implementing our fractional orthotriple fuzzy figures with the situation $0 \leq P^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) + N^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) + I^{\breve{g}}_{\breve{A}_{\hat{s}}}(\acute{r}) \leq 1$, then the exponential operation on fractional orthotriple fuzzy numbers provides values in the prolonged range $[1,2]$ with the constraint that $1 \leq (\epsilon^{\sqrt[\breve{g}]{1-^{P^{\breve{g}}_{\breve{A}_{\hat{s}}}}}})^{\breve{g}} + (\sqrt[\breve{g}]{1-\epsilon^{\breve{g}N_{\breve{A}_{\hat{s}}}}})^{\breve{g}} + (\sqrt[\breve{g}]{1-\epsilon^{\breve{g}I_{\breve{A}_{\hat{s}}}}})^{\breve{g}} \leq 2$.

Also, new AOs such as the FOFWEA, FOFOWEA, FOFHWEA, FOFEWEA, FOFEOWEA, and FOFEHWEA were presented in a fractional orthotriple fuzzy environment and their features were investigated. Eventually, the suitability and the value of the suggested AOs were shown by applying a real-world example.

Further, some new aggregation operators, such as Einstein geometric aggregation operator, Bonferroni mean aggregation operator, and Yager ordered weighted average (OWA) aggregation operators, will be established in the future for the FOFS models, applying the exponential operational principles and Einstein exponential operational principles.

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